Applications of Krein's theory of regular symmetric operators to sampling theory

2007 J. Phys. A: Math. Theor. 40 9413

(http://iopscience.iop.org/1751-8121/40/31/017)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9413–9426 doi:10.1088/1751-8113/40/31/017

Applications of Krein's theory of regular symmetric operators to sampling theory*

Luis O Silva and Julio H Toloza

Departamento de Métodos Matemáticos y Numéricos, Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, C.P. 04510 México D.F., Mexico

E-mail: silva@leibniz.iimas.unam.mx and jtoloza@leibniz.iimas.unam.mx

Received 22 February 2007, in final form 11 June 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9413

Abstract

The classical Kramer sampling theorem establishes general conditions that allow the reconstruction of functions by mean of orthogonal sampling formulae. One major task in sampling theory is to find concrete, non-trivial realizations of this theorem. In this paper, we provide a new approach to this subject on the basis of Krein's theory of representation of simple regular symmetric operators having deficiency indices (1, 1). We show that the resulting sampling formulae have the form of Lagrange interpolation series. We also characterize the space of function reconstructible by our sampling formulae. Our construction allows a rigorous treatment of certain ideas proposed recently in quantum gravity.

PACS numbers: 02.30.Tb, 02.60.Ed, 02.70.Hm, 04.60.−m, 04.62.+v, 89.70.+c
Mathematics Subject Classification: 41A05, 46E22, 47B25, 47N50, 47N99, 94A20

1. Introduction

It has been argued recently that sampling theory might be the bridge that would allow to reconcile the continuous nature of physical fields with the need of discretization of spacetime, as required by a yet-to-be-formulated theory of quantum gravity [14–16]. This idea, which is partially developed in [14] for the one-dimensional case, introduces the use of simple regular symmetric operators with deficiency indices (1, 1) to obtain orthogonal sampling formulae. It is remarkable that the class of operators under consideration in [14] had been already studied in detail by Krein [19–21] more than 60 years ago.

The conjunction of the ideas of [14] and Krein's theory of symmetric operators with equal deficiency indices suggests the means of treating sampling theory in a new and straightforward

* Research partially supported by CONACYT under project P42553F.

way. By introducing this new approach, we put in a mathematically rigorous framework the ideas in [14] and propose a general method for obtaining analytic sampling formulae associated with the self-adjoint extensions of a simple regular symmetric operator with deficiency indices (1, 1).

A seminal result in sampling theory is the Whittaker–Shannon–Kotel'nikov (WSK) sampling theorem [17, 25, 27]. This theorem states that functions that belong to a Paley–Wiener space may be uniquely reconstructed from their values at certain discrete sets of points. A general approach to WSK-type formulae was developed by Kramer [18] based on the following result. Given a finite interval $I = [a, b]$, let $K(y, x) \in L^2(I, \mathrm{d}y)$ for all $x \in \mathbb{R}$. Assume that there exists a sequence $\{x_n\}_{n\in\mathbb{Z}}$ for which $\{K(y, x_n)\}_{n\in\mathbb{Z}}$ forms an orthogonal basis of $L^2(I, \mathrm{d}y)$. Let f be any function of the form

$$f(x) := \langle K(\cdot, x), g(\cdot)\rangle_{L^2(I)}$$

for some $g \in L^2(I, \mathrm{d}y)$, where the brackets denote the inner product in $L^2(I, \mathrm{d}y)$. Then f can be reconstructed from its samples $\{f(x_n)\}_{n\in\mathbb{Z}}$ by the formula

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\langle K(\cdot, x), K(\cdot, x_n)\rangle_{L^2(I)}}{\|K(\cdot, x_n)\|^2_{L^2(I)}} f(x_n), \qquad x \in \mathbb{R}.$$

The search for concrete realizations of the Kramer sampling theorem, which in some cases has proven to be a difficult task, has motivated a large amount of literature. See, for instance, [3, 9–11, 28, 29] and references therein. On the basis of the approach proposed in the present work, we obtain analytic sampling formulae for functions belonging to linear sets of analytic functions determined by a given regular simple symmetric operator.

A central idea in Krein's theory is the fact that any simple symmetric operator with deficiency indices (1, 1), acting in a certain Hilbert space $\mathcal{H}$, defines a bijective mapping from $\mathcal{H}$ onto a space of scalar functions of one complex variable having certain analytic properties. In this space of functions one can introduce an inner product. In the particular case when the starting point is an entire operator (see definition 2), the space of functions turns out to be a Hilbert space of entire functions known as a de Branges space [5].

Summing up, in this paper we present an original technique in sampling theory based on Krein's theory of regular symmetric operators. Concretely, we obtain an analytic sampling formula valid for functions belonging to linear sets of analytic functions associated with symmetric operators of the kind already mentioned. We then focus our attention on entire operators to provide a characterization of the corresponding spaces of entire functions. As a byproduct, we also provide rigorous proofs to some of the results formally obtained in [14]. In a sense this work may be considered as introductory to the theoretical framework. Further rigorous results related to [14], as well as other applications to sampling theory and inverse spectral theory, will be discussed in subsequent papers.

Our approach to sampling theory allows us to reconstruct functions that are out of the scope of certain techniques developed recently [9]. It is worth remarking that, in terms of the method derived in this paper, the results of [9] correspond to considering only a particular self-adjoint extension of an entire operator. We also remark that [9] does not fit in the framework proposed in [14] for which, in contrast, our results indeed give some rigorous justification.

This paper is organized as follows. In section 2, we introduce some concepts of the theory of regular symmetric operators. In section 3, we state and prove a sampling formula. In section 4, we provide a characterization of the Hilbert spaces associated with the class of operators under consideration, paying particular attention to the case of entire operators. In section 5, we discuss some examples. Finally, we discuss some ideas for further investigation in section 6.

2. Preliminaries

Most of the mathematical background required in this work is based on Krein's theory of representation of symmetric operators as accounted in the expository book [12]. In this section we review some definitions and results from the theory of symmetric operators and introduce the notation.

Let $\mathcal{H}$ denote a separable Hilbert space whose inner product $\langle \cdot, \cdot \rangle$ will be assumed anti-linear in its first argument.

Definition 1. *A closed symmetric operator A with domain and range in $\mathcal{H}$ is called:*

(a) simple if it does not have non-trivial invariant subspaces on which A is self-adjoint,
(b) regular if every point in $\mathbb{C}$ is a point of regular type for A, that is, if the operator $(A - zI)$ has bounded inverse for every $z \in \mathbb{C}$.

By [12, theorem 1.2.1], an operator A is simple if and only if

$$\bigcap_{z:\mathrm{Im}\, z\neq 0} \mathrm{Ran}(A - zI) = \{0\}.$$

Moreover, as shown in [12, propositions 1.3.3, 1.3.5 and 1.3.6], a simple closed symmetric operator with deficiency indices (1, 1) is regular if and only if some (hence every) self-adjoint extension of A within $\mathcal{H}$ has only a discrete spectrum of multiplicity 1. Also,

$$\dim[\mathrm{Ran}(A - zI)]^{\perp} = \dim[\mathrm{Ker}(A^* - \overline{z}I)] = 1 \tag{2.1}$$

for all $z \in \mathbb{C}$. For this kind of operators it is also known that, for any given $x \in \mathbb{R}$, there is exactly one self-adjoint extension within $\mathcal{H}$ such that x is in its spectrum [12, proposition 1.3.5].

Remark 1. In what follows, whenever we consider self-adjoint extensions of a symmetric operator A, we will always mean the self-adjoint restrictions of A^*, in other words, the self-adjoint extensions of A within $\mathcal{H}$ (cf Naimark's theory on generalized self-adjoint extensions [2, appendix 1]).

We will denote by $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ the class of regular, simple, closed symmetric operators, defined on $\mathcal{H}$, with deficiency indices (1, 1).

Let $A_\sharp$ be some self-adjoint extension of $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$. The generalized Cayley transform is defined by

$$(A_\sharp - wI)(A_\sharp - zI)^{-1}$$

for every $w, z \in \mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$. This operator has several properties [12, p 9]. We only mention the following one:

$$(A_\sharp - wI)(A_\sharp - zI)^{-1} : \mathrm{Ker}(A^* - wI) \to \mathrm{Ker}(A^* - zI) \tag{2.2}$$

one to one and onto.

A complex conjugation on $\mathcal{H}$ is a bijective anti-linear operator $C : \mathcal{H} \to \mathcal{H}$ such that $C^2 = I$ and $\langle C\eta, C\varphi\rangle = \langle \varphi, \eta\rangle$ for all $\eta, \varphi \in \mathcal{H}$. A symmetric operator A is said to be real with respect to a complex conjugation C if $C\,\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A)$ and $CA\varphi = AC\varphi$ for every $\varphi \in \mathrm{Dom}(A)$. Clearly, the condition $C\,\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A)$, along with $C^2 = I$, implies that $C\,\mathrm{Dom}(A) = \mathrm{Dom}(A)$. If moreover A has deficiency indices (1, 1), then A^* is also real with respect to C (see the proof of [26, corollary 2.5]).

3. Sampling formulae

Let us consider an operator $A \in \mathrm{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$. Let $A_\sharp$ be some self-adjoint extension of A. Given $z_0 \in \mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$ and $\psi_0 \in \mathrm{Ker}(A^* - z_0 I)$, define

$$\psi(z) := (A_\sharp - z_0 I)(A_\sharp - zI)^{-1}\psi_0 = \psi_0 + (z - z_0)(A_\sharp - zI)^{-1}\psi_0 \tag{3.1}$$

for every $z \in \mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$. This vector-valued function is analytic in the resolvent set of $A_\sharp$ and, by (2.2), takes values in $\mathrm{Ker}(A^* - zI)$ when evaluated at z. Moreover, $\psi(z)$ has simple poles at the points of $\mathrm{Sp}(A_\sharp) = \mathrm{Sp}_{\mathrm{disc}}(A_\sharp)$. By looking at the first equality in (3.1), it is clear that the dependence of $\psi(z)$ on z_0 and ψ_0 is rather unessential. Indeed, for any other $z_0' \in \mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$, we can take $\psi_0' = (A_\sharp - z_0 I)(A_\sharp - z_0' I)^{-1}\psi_0$ in $\mathrm{Ker}(A^* - z_0' I)$ and thus $\psi(z) = \psi_0' + (z - z_0')(A_\sharp - zI)^{-1}\psi_0'$ by the first resolvent identity.

Given $z_1 \in \mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$, let us choose some $\mu \in \mathcal{H}$ such that $\langle\psi(\overline{z_1}), \mu\rangle \neq 0$. The inner product $\langle\psi(\overline{z}), \mu\rangle$ then defines an analytic function in $\mathbb{C}\backslash\mathrm{Sp}(A_\sharp)$, having zeros at a countable set S_μ devoid of accumulation points in $\mathbb{C}$. The set S_μ is defined as the subset of $\mathbb{C}$ for which $\mathcal{H}$ cannot be written as the direct sum of $\mathrm{Ran}(A - zI)$ and $\mathrm{Span}\{\mu\}$. Following Krein, we call the element μ a *gauge* [21].

Let us define

$$\xi(z) := \frac{\psi(\overline{z})}{\langle\mu, \psi(\overline{z})\rangle} \tag{3.2}$$

for all $z \in \mathbb{C}\backslash S_\mu$.

Lemma 1. *The vector-valued function $\xi(z)$ is anti-analytic in $\mathbb{C}\backslash S_\mu$ and has simple poles at points of S_μ. Moreover, it does not depend on the self-adjoint extension of A used to define $\psi(z)$.*

Proof. The first statement holds by rather obvious reasons, note only that the poles of $\langle\psi(\overline{z}), \mu\rangle$ coincide with those of $\psi(\overline{z})$. Let us pay attention to the second statement. Consider two different self-adjoint extensions $A_\sharp$ and $A_\sharp'$. We have

$$\psi(z) = (A_\sharp - z_0 I)(A_\sharp - zI)^{-1}\psi_0 \qquad \text{and} \qquad \psi'(z) = (A_\sharp' - z_0 I)(A_\sharp' - zI)^{-1}\psi_0.$$

Since both $\psi(z)$ and $\psi'(z)$ belong to $\mathrm{Ker}(A^* - zI)$ and the dimension of this subspace is always equal to one, it follows that $\psi'(z) = g(z)\psi(z)$ for every $z \notin \mathrm{Sp}(A_\sharp) \cup \mathrm{Sp}(A_\sharp')$, where $g(z)$ is a scalar function. Inserting this identity into (3.2) yields $\xi(z) = \xi'(z)$. □

For every $z \in \mathbb{C}\backslash S_\mu$, we have the decomposition $\mathcal{H} = \mathrm{Ran}(A - zI) \dotplus \mathrm{Span}\{\mu\}$, in which case every element $\varphi \in \mathcal{H}$ can be written as

$$\varphi = [\varphi - \widehat{\varphi}(z)\mu] + \widehat{\varphi}(z)\mu,$$

where $\varphi - \widehat{\varphi}(z)\mu \in \mathrm{Ran}(A - zI)$. A simple computation shows that the non-orthogonal projection $\widehat{\varphi}(z)$ is given by

$$\widehat{\varphi}(z) := \frac{\langle\psi(\overline{z}), \varphi\rangle}{\langle\psi(\overline{z}), \mu\rangle} = \langle\xi(z), \varphi\rangle, \tag{3.3}$$

whenever $z \in \mathbb{C}\backslash S_\mu$; it is otherwise not defined. Indeed, because of the anti-linearity of the inner product in its first argument, the function $\widehat{\varphi}(z)$ is analytic in $\mathbb{C}\backslash S_\mu$ and meromorphic in $\mathbb{C}$ for every $\varphi \in \mathcal{H}$. We note that $\widehat{\mu}(z) \equiv 1$.

Let us denote the linear map $\varphi \mapsto \widehat{\varphi}(z)$ by Φ_μ and the linear space of functions given by (3.3) by $\Phi_\mu\mathcal{H}$. Since the operator A is simple, it follows that Φ_μ is injective and, therefore, is an isomorphism from $\mathcal{H}$ onto $\Phi_\mu\mathcal{H}$ [12, theorem 1.2.2]. Moreover, Φ_μ transforms A into the multiplication operator on $\Phi_\mu\mathcal{H}$, that is,

$$\widehat{(A\varphi)}(z) = \left(\Phi_\mu A \Phi_\mu^{-1}\widehat{\varphi}\right)(z) = z\widehat{\varphi}(z), \qquad \varphi \in \mathrm{Dom}(A).$$

Proposition 1. *Assume $S_\mu \cap \mathbb{R} = \emptyset$. Let $\{x_n\}$ be the spectrum of any self-adjoint extension $A_\sharp$ of A. Then, for any analytic function $f(z)$ that belongs to $\Phi_\mu\mathcal{H}$, we have*

$$f(z) = \sum_{x_n \in \mathrm{Sp}(A_\sharp)} \frac{\langle \xi(z), \xi(x_n)\rangle}{\|\xi(x_n)\|^2} f(x_n). \tag{3.4}$$

The convergence in (3.4) is uniform over compact subsets of $\mathbb{C}\backslash S_\mu$.

Proof. Fix some arbitrary self-adjoint extension $A_\sharp$ of A. Take another self-adjoint extension $A'_\sharp \neq A_\sharp$ to define $\psi(z)$, that is, $\psi(z) = (A'_\sharp - z_0 I)(A'_\sharp - zI)^{-1}\psi_0$. Arrange the elements of $\mathrm{Sp}(A_\sharp)$ in a sequence $\{x_n\}_{n\in M}$, where M is a countable indexing set, and let η_n be an eigenstate of $A_\sharp$ corresponding to x_n, i.e., $A_\sharp\eta_n = x_n\eta_n$. Since $A^* \supset A_\sharp$, it follows that $\eta_m \in \mathrm{Ker}(A^* - x_m I)$, where furthermore $\dim[\mathrm{Ker}(A^* - x_m I)] = 1$. On the other hand, since x_m is not a pole of $\psi(z)$, $\psi(x_m)$ is well defined and belongs also to $\mathrm{Ker}(A^* - x_m I)$. Therefore, up to a factor, $\eta_m = \xi(x_m)$.

Pick a sequence $\{M_k\}_{k\in\mathbb{N}}$ of subsets of M such that $M_k \subset M_{k+1}$ and $\cup_k M_k = M$. Consider any analytic function $f(z) \in \Phi_\mu\mathcal{H}$. Since Φ_μ is injective, there exists a unique $\varphi \in \mathcal{H}$ such that $\widehat{\varphi}(z) = f(z)$. Clearly,

$$\left|\widehat{\varphi}(z) - \sum_{n\in M_k} \frac{\langle \xi(z), \xi(x_n)\rangle}{\|\xi(x_n)\|^2}\langle \xi(x_n), \varphi\rangle\right| \leqslant \|\xi(z)\| \left\|\varphi - \sum_{n\in M_k} \frac{1}{\|\xi(x_n)\|^2}\langle \xi(x_n), \varphi\rangle \xi(x_n)\right\|.$$

The second factor in the rhs of this inequality does not depend on z and obviously tends to zero as $k \to \infty$. Since $\xi(z)$ is continuous on $\mathbb{C}\backslash S_\mu$, the uniform convergence of (3.4) has been proven. □

Remark 2. Krein asserts that, for any operator in $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, one can always choose μ so that $S_\mu \cap \mathbb{R} = \emptyset$ [20, theorem 8].

Clearly, the sequence $\{x_n\}$ could be replaced by any other sequence $\{z_n\}$ of complex numbers for which $\{\xi(z_n)\}$ is an orthogonal basis of $\mathcal{H}$ (for a related discussion, see [10]). In our case, the question of whether such a sequence exists or not is answered to the affirmative by invoking the self-adjoint extensions of the operator A. Note also that, since for any real number x there is a self-adjoint extension of A containing x in its spectrum, it follows that every real point can be taken as a sampling point. In contrast with this, the construction of [9] has a fixed set of sampling points.

Below we show that the interpolation formula (3.4) is indeed a Lagrange interpolation series.

Proposition 2. *Under the hypotheses of proposition 1, there exists a complex function $G(z)$, analytic in $\mathbb{C}\backslash S_\mu$ (hence $\mathbb{R}$) and having simple zeros at $\mathrm{Sp}(A_\sharp)$, such that*

$$f(z) = \sum_{x_n \in \mathrm{Sp}(A_\sharp)} \frac{G(z)}{(z - x_n)G'(x_n)} f(x_n),$$

for every $f(z) \in \Phi_\mu\mathcal{H}$.

Proof. Given $\{x_n\}_{n\in M} = \mathrm{Sp}(A_\sharp)$, set $\psi(z) = (A_\sharp - z_0 I)(A_\sharp - zI)^{-1}\psi_0$ for some z_0 such that $\operatorname{Im} z_0 \neq 0$ and $\psi_0 \in \operatorname{Ker}(A^* - z_0 I)$. Define

$$G(z) := \frac{1}{\langle \psi(\overline{z}), \mu\rangle}.$$

This function has simple zeros at the poles of $\psi(\overline{z})$, that is, at points of the set $\{x_n\}_{n\in M}$. Also, $\xi(z) = \overline{G(z)}\psi(\overline{z})$. Moreover, we can write

$$\psi(\overline{z}) = \frac{\eta(\overline{z})}{\overline{z} - x_n} \qquad \text{and} \qquad G(z) = (\overline{z} - x_n)F(\overline{z}),$$

where $\eta(z) := (z - x_n)(A_\sharp - z_0 I)(A_\sharp - zI)^{-1}\psi_0$ is analytic at $z = x_n$, $\eta(x_n) \neq 0$ and $F(x_n) = G'(x_n)$. Thus, a straightforward computation shows that

$$\frac{\langle \xi(z), \xi(x_n)\rangle}{\|\xi(x_n)\|^2} = \frac{G(z)}{(z - x_n)G'(x_n)} \frac{\langle \eta(\overline{z}), \eta(x_n)\rangle}{\|\eta(x_n)\|^2},$$

so it remains to verify that the last factor above equals one. By the Cauchy integral formula, we have

$$\begin{aligned}
\eta(x_n) &= \frac{1}{2\pi \mathrm{i}} \oint_{|w - x_n| = \epsilon} \frac{\eta(w)}{w - x_n}\,\mathrm{d}w \\
&= \frac{1}{2\pi \mathrm{i}} \oint_{|w - x_n| = \epsilon} (A_\sharp - z_0 I)(A_\sharp - wI)^{-1}\psi_0\,\mathrm{d}w \\
&= -(A_\sharp - z_0 I)\left(\frac{1}{2\pi \mathrm{i}} \oint_{|w - x_n| = \epsilon} (wI - A_\sharp)^{-1}\,\mathrm{d}w\right)\psi_0 \\
&= -(x_n - z_0)P_n\psi_0,
\end{aligned}$$

where P_n denotes the orthoprojector onto the eigenspace associated with x_n. Therefore,

$$\begin{aligned}
\langle \eta(\overline{z}), \eta(x_n)\rangle &= -(x_n - z_0)\langle \eta(\overline{z}), P_n\psi_0\rangle \\
&= -(x_n - z_0)(z - x_n)\langle P_n(A_\sharp - z_0 I)(A_\sharp - \overline{z}I)^{-1}\psi_0, \psi_0\rangle \\
&= |x_n - z_0|^2\langle P_n\psi_0, \psi_0\rangle.
\end{aligned}$$

Finally, it is clear that $\langle \eta(x_n), \eta(x_n)\rangle = |x_n - z_0|^2\langle P_n\psi_0, \psi_0\rangle$. □

Remark 3. Note that the function $G(z)$ is defined up to a constant factor. In particular, one may adjust it so that $G'(x_k) = 1$, where x_k is a fixed eigenvalue of $A_\sharp$. Thus, a computation like that in the proof above shows that

$$G(z) = (z - x_k)\frac{\langle \xi(z), \xi(x_k)\rangle}{\|\xi(x_k)\|^2}. \tag{3.5}$$

This identity may be useful in some applications; see example 2.

4. Spaces of analytic functions

In this section we characterize the set of functions given by the mapping Φ_μ. Note that in the general case $\Phi_\mu\mathcal{H}$ is a space of analytic functions with simples poles in a subset of S_μ.

Let $\mathcal{R} \subset \mathcal{H}$ be the linear space of elements φ for which $\widehat{\varphi}(z)$ is analytic on $\mathbb{R}$. As a consequence of [12, corollary 1.2.1], it follows that

$$\langle \varphi, \eta\rangle = \int_{-\infty}^{\infty} \overline{\widehat{\varphi}(x)}\widehat{\eta}(x)\,\mathrm{d}m(x) \tag{4.1}$$

for any $\varphi, \eta \in \mathcal{R}$ and $m(x) = \langle E_x\mu, \mu\rangle$, where E_x is any spectral function of the operator A. That is, $\Phi_\mu\mathcal{R}$ is a linear space of analytic functions in $\mathbb{C}\backslash S_\mu$ such that their restriction to $\mathbb{R}$ belong to $L^2(\mathbb{R}, \mathrm{d}m)$; in short,

$$\Phi_\mu\mathcal{R}|_\mathbb{R} \subset L^2(\mathbb{R}, \mathrm{d}m).$$

Moreover, in this restricted sense Φ_μ is an isometry from $\mathcal{R}$ into $L^2(\mathbb{R}, \mathrm{d}m)$.

The following theorem is due to Krein [20, theorem 3].

Theorem 1 (Krein). *For $A \in \mathrm{Sym}_\mathrm{R}^{(1,1)}(\mathcal{H})$, assume that $\overline{\mathcal{R}} = \mathcal{H}$. Consider a distribution function $m(x) = \langle E_x\mu, \mu\rangle$, where E_x is a spectral function of A. Then the map Φ_μ generates a bijective isometry from $\mathcal{H}$ onto $L^2(\mathbb{R}, \mathrm{d}m)$ if and only if E_x is orthogonal.*

This theorem deserves some comments. When E_x is orthogonal it occurs that

$$\Phi_\mu\mathcal{H}|_\mathbb{R} = L^2(\mathbb{R}, \mathrm{d}m)$$

in the usual sense of equivalence classes. Thus, every function in $L^2(\mathbb{R}, \mathrm{d}m)$ is, up to a set of measure zero with respect to $m(x)$, the restriction to $\mathbb{R}$ of a unique function that is the image under Φ_μ of one and only one element belonging to $\mathcal{H}$. Note that any orthogonal spectral function of A corresponds to the spectral function of one of its self-adjoint extensions within $\mathcal{H}$. Since these self-adjoint extensions have only a discrete spectrum, the inner product in $L^2(\mathbb{R}, \mathrm{d}m)$, with $m(x) = \langle E_x\mu, \mu\rangle$, reduces to an expression like

$$\int_{-\infty}^{\infty} \overline{f(x)}g(x)\,\mathrm{d}m(x) = \sum_k c_k\overline{f(x_k)}g(x_k)$$

whenever E_x is orthogonal. That is, the equivalence classes in these spaces are quite broad.

The following corollary is partly a straightforward consequence of theorem 1. Note that $S_\mu \cap \mathbb{R} = \emptyset$ implies $\mathcal{R} = \mathcal{H}$.

Corollary 1. *Let $A \in \mathrm{Sym}_\mathrm{R}^{(1,1)}(\mathcal{H})$ and choose a gauge μ for this operator. Assume that $S_\mu \cap \mathbb{R} = \emptyset$. Let E_x be one of its orthogonal spectral functions. Then the linear space of functions $\widehat{\mathcal{H}}_\mu := \Phi_\mu\mathcal{H}$, equipped with the inner product*

$$\langle f(\cdot), g(\cdot)\rangle := \int_{-\infty}^{\infty} \overline{f(x)}g(x)\,\mathrm{d}m(x) \qquad \text{where} \quad m(x) = \langle E_x\mu, \mu\rangle, \tag{4.2}$$

is a reproducing kernel Hilbert space, with reproducing kernel $k(z, w) := \langle \xi(z), \xi(w)\rangle$.

Proof. We only verify the last statement. Given $f(z) = \langle \xi(z), \varphi\rangle \in \widehat{\mathcal{H}}_\mu$, we have

$$\langle k(\cdot, w), f(\cdot)\rangle = \int_{-\infty}^{\infty} \overline{k(x, w)}f(x)\,\mathrm{d}m(x) = \langle \xi(w), \varphi\rangle = f(w),$$

where the second equality follows from (4.1). □

Note that, once the operator A is given, the linear space $\widehat{\mathcal{H}}_\mu$ depends only on the choice of gauge μ. By corollary 1, for those gauges that obeys $S_\mu \cap \mathbb{R} = \emptyset$, $\widehat{\mathcal{H}}_\mu$ may be endowed with different Hilbert space structures, one for each orthogonal spectral function of the operator A. By (4.1), all these Hilbert spaces are however isometrically equivalent.

Irrespective of anyone of these Hilbert space structures, $\widehat{\mathcal{H}}_\mu$ possesses the following properties.

Proposition 3. *Suppose μ such that $S_\mu \cap \mathbb{R} = \emptyset$ and consider $\widehat{\mathcal{H}}_\mu$ equipped with an inner product of the form (4.2).*

(i) Let w be a non-real zero of $f(z) \in \widehat{\mathcal{H}}_\mu$. Then the function $g(z) := f(z)(z-\overline{w})(z-w)^{-1}$ is also in $\widehat{\mathcal{H}}_\mu$ and $\|g(\cdot)\| = \|f(\cdot)\|$.
(ii) The evaluation functional, defined by $f(\cdot) \mapsto f(z)$, is continuous.

Proof. (i) Suppose that w is a non-real zero of $f(z)$. Since $f(z) = \langle \xi(z), \varphi\rangle$ for some $\varphi \in \mathcal{H}$, it follows that φ is orthogonal to $\psi(\overline{w})$ and therefore $\varphi \in \operatorname{Ran}(A - wI)$. Note that

$$f(z) = \langle \xi(z), (A - wI)(A - wI)^{-1}\varphi\rangle = (z - w)\langle \xi(z), (A - wI)^{-1}\varphi\rangle.$$

In these computations we have used that $\xi(z) \in \operatorname{Ker}(A^* - \overline{z})$. Moreover,

$$\langle \xi(z), (A - \overline{w}I)(A - wI)^{-1}\varphi\rangle = (z - \overline{w})\langle \xi(z), (A - wI)^{-1}\varphi\rangle = \frac{z - \overline{w}}{z - w} f(z).$$

Then $f(z)(z - \overline{w})(z - w)^{-1} \in \widehat{\mathcal{H}}_\mu$. The equality of norms follows from the fact that the Cayley transform $(A_\sharp - \overline{w}I)(A_\sharp - wI)^{-1}$ is an isometry.

(ii) Let $f(z), g(z) \in \widehat{\mathcal{H}}_\mu$. Then $f(z) = \langle \xi(z), \varphi\rangle$ and $g(z) = \langle \xi(z), \eta\rangle$ for some $\varphi, \eta \in \mathcal{H}$, and furthermore

$$|f(w) - g(w)| = |\langle \xi(w), \varphi - \eta\rangle| \leqslant \|\xi(w)\|\|\varphi - \eta\| = \|\xi(w)\|\|f(\cdot) - g(\cdot)\|.$$

In other words, this result follows from the fact that $\widehat{\mathcal{H}}_\mu$ is a reproducing kernel Hilbert space. □

Definition 2. *An operator $A \in \operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$ is called entire if there exists a so-called entire gauge $\mu \in \mathcal{H}$ such that $\widehat{\varphi}(z)$ is an entire function for every $\varphi \in \mathcal{H}$. Equivalently, A is entire if $\mathcal{H} = \operatorname{Ran}(A - zI) \dotplus \operatorname{Span}\{\mu\}$ for all $z \in \mathbb{C}$.*

Note that if A is entire, $\xi(z)$ is a vector-valued (anti)-entire function and $\widehat{\mathcal{H}}_\mu$ is a Hilbert space of entire functions. The spaces of reconstructible functions treated in [9] correspond to a special situation within this particular case of our approach.

Definition 3. *A Hilbert space of entire functions is called a de Branges space if, for every $f(z)$ in that space, the following conditions holds:*

(i) for every non-real zero w of $f(z)$, the function $f(z)(z - \overline{w})(z - w)^{-1}$ belongs to the Hilbert space and has the same norm as $f(z)$,
(ii) the function $f^*(z) := \overline{f(\overline{z})}$ belongs to the Hilbert space and also has the same norm as $f(z)$; and furthermore
(iii) for every $w : \operatorname{Im} w \neq 0$, the linear functional $f(\cdot) \mapsto f(w)$ is continuous.

There is an extensive literature concerning the properties of de Branges spaces. We refer to [5] for more details.

It can be shown that $\widehat{\mathcal{H}}_\mu$ is a de Branges space for certain choices of the entire gauge μ. There are some evidence indicating that Krein noted this fact [12, p 209]. Also, some hints supporting this assertion have been given by de Branges himself [6]. We however could not find any formal proof of this statement. Thus, for the sake of completeness and for the lack of a proper reference, we provide a proof below.

Remark 4. As a consequence of [12, lemma 2.7.1], given any self-adjoint extension $A_\sharp$ of an operator $A \in \operatorname{Sym}_{\mathbb{R}}^{(1,1)}(\mathcal{H})$, one can always find a complex conjugation C for which $A_\sharp$ is real. If follows from the proof of the cited lemma that $C\psi(z) = \psi(\overline{z})$ when $\psi(z)$ is written in terms of the real self-adjoint extension. Moreover, by [12, theorem 2.7.1], the operator A

is also real with respect to C. Since by [26, corollary 2.5] all the self-adjoint extensions of A are real, it follows that $C\psi(z) = \psi(\overline{z})$ for every realization of $\psi(z)$.

If furthermore A is entire, then an entire gauge μ may be chosen real, that is, $C\mu = \mu$ (see [19, theorem 1] and also [12, section 2.7.7]).

Proposition 4. *Assume that an entire operator A is real with respect to some complex conjugation C and let μ be a real entire gauge. Then the associated Hilbert space $\widehat{\mathcal{H}}_\mu$ is a de Branges space.*

Proof. In view of proposition 3, we only have to verify (ii). By remark 4 we know that $C\psi(\overline{z}) = \psi(z)$ thence $C\xi(z) = \xi(\overline{z})$. Now consider any $f(z) = \langle \xi(z), \varphi \rangle$. Clearly $f^*(z) := \langle \xi(z), C\varphi \rangle$ also belongs to $\widehat{\mathcal{H}}_\mu$. Furthermore,

$$f^*(z) = \langle \xi(z), C\varphi \rangle = \overline{\langle C\xi(z), \varphi \rangle} = \overline{\langle \xi(\overline{z}), \varphi \rangle} = \overline{f(\overline{z})}.$$

Since C is an isometry, the equality of norms follows. □

Note that we only have used the simultaneous reality of the entire operator and its entire gauge μ in showing that $\widehat{\mathcal{H}}_\mu$ obeys (ii) of definition 3. Indeed, this condition is also necessary.

Proposition 5. *If $\widehat{\mathcal{H}}_\mu$ is a de Branges space there is a complex conjugation C with respect to which both A and μ are real.*

Proof. Let $\widehat{C} : \widehat{\mathcal{H}}_\mu \to \widehat{\mathcal{H}}_\mu$ be defined by $(\widehat{C}f)(z) = \overline{f(\overline{z})}$, for every $f(z) \in \widehat{\mathcal{H}}_\mu$. Clearly, $\widehat{C}$ is a complex conjugation. Moreover, the multiplication operator $\widehat{A}$, defined on the maximal domain in $\widehat{\mathcal{H}}_\mu$ by $(\widehat{A}f)(z) = zf(z)$, is real with respect to $\widehat{C}$. Now define $C := \Phi_\mu^{-1}\widehat{C}\Phi_\mu$. By construction, C is a complex conjugation in $\mathcal{H}_\mu$. Since $(\widehat{C}\widehat{\mu})(z) \equiv 1 \equiv \widehat{\mu}(z)$, it follows that μ is real with respect to C. Finally, it is not difficult to see that $A = \Phi_\mu^{-1}\widehat{A}\Phi_\mu$ and therefore A is real with respect to C. □

An alternative definition of a de Branges space is given in terms of an arbitrary entire function $e(z)$. The de Branges space $\mathcal{H}(e)$ associated with $e(z)$ is

$$\mathcal{H}(e) = \left\{ f(z) \in \mathrm{Hol}(\mathbb{C}) : f(z)/e(z) \in \mathcal{N},\ f^*(z)/e(z) \in \mathcal{N},\ \int_{-\infty}^{\infty} |f(x)/e(x)|^2\, \mathrm{d}x < \infty \right\},$$

where $\mathcal{N}$ denotes the class of analytic functions of bounded type and non-positive mean type in the upper half plane [5, chapter 1]. From this definition, it follows that $\mathcal{H}(e)$ is a reproducing kernel Hilbert space, whose reproducing kernel is expressed in terms of the function $e(z)$. As shown in [5, chapter 2] (see also [13, section 5]) a Hilbert space of entire functions $\tilde{\mathcal{H}}$ that obeys definition 3 is unitarily equivalent to a de Branges space $\mathcal{H}(e)$ with

$$e(z) = \mathrm{i}\sqrt{\frac{\pi}{k(w_0, w_0)\,\mathrm{Im}(w_0)}}\,(\overline{w_0} - z)k(w_0, z),$$

where $k(w, z)$ is the reproducing kernel of $\tilde{\mathcal{H}}$ and w_0 is any non-real complex number such that $k(w_0, w_0) > 0$.

As discussed in [5, 13], the multiplication operator $\widehat{A}$ with maximal domain in $\mathcal{H}(e)$ is a closed, symmetric operator with deficiency indices $(1, 1)$ and domain of codimension 0 or 1. Since in our case the multiplication operator is unitarily equivalent to an operator belonging to $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$, the codimension of $\mathrm{Dom}(\widehat{A})$ is necessarily equal to 0, that is, $\widehat{A}$ is densely defined. The set of self-adjoint extensions of $\widehat{A}$ is in one–one correspondence with the set of entire functions $s_t(z) := -\sin t a(z) + \cos t b(z)$, $t \in [0, \pi)$, where $a(z)$ and $b(z)$ are defined by the identity $e(z) = a(z) + \mathrm{i}b(z)$. In terms of $s_t(z)$, we have

$$\mathrm{Dom}(\widehat{A}_t) = \left\{ g(z) = \frac{s_t(w_0) f(z) - s_t(z) f(w_0)}{z - w_0} : f(z) \in \mathcal{H}(e),\ \text{fixed } w_0 : s_t(w_0) \neq 0 \right\},$$

$$\widehat{A}_t g(z) = z g(z) + f(w_0) s_t(z).$$

For details, see [13, section 6]. Note that the definition of $\mathrm{Dom}(\widehat{A}_t)$ does not depend on the choice of w_0, as one can verify by resorting to the first resolvent identity.

5. Examples

In this section, we give two elementary illustrations of the method developed above. It is worth mentioning that one can find particular realizations of our general approach in previous works on sampling theory, which do not make any connection to Krein's theory of representation of operators as developed here. In particular, a sampling theory has been obtained from Krein's theory of string operators [4].

(1) Consider the semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix}, \tag{5.1}$$

where $b_k > 0$ and $q_k \in \mathbb{R}$ for $k \in \mathbb{N}$. Fix an orthonormal basis $\{\delta_k\}_{k\in\mathbb{N}}$ in $\mathcal{H}$. Let J be the operator in $\mathcal{H}$ whose matrix representation with respect to $\{\delta_k\}_{k\in\mathbb{N}}$ is (5.1). Thus, J is the minimal closed operator satisfying

$$\langle \delta_n, J\delta_n \rangle = q_n, \qquad \langle \delta_{n+1}, J\delta_n \rangle = \langle \delta_n, J\delta_{n+1} \rangle = b_n, \qquad \forall n \in \mathbb{N}.$$

(Consult [2, section 47] for a discussion on matrix representation of unbounded symmetric operators.) It is well known that J may have either deficiency indices $(1, 1)$ or $(0, 0)$ [1, chapter 4, section 1.2]. A classical result is that if J has deficiency indices $(1, 1)$, then the orthogonal polynomials of the first kind $P_k(z)$ associated with (5.1) are such that

$$\sum_{k=0}^{\infty} |P_k(z)|^2 < \infty$$

uniformly in any compact domain of the complex plane [1, theorem 1.3.2]. Therefore, for any $z \in \mathbb{C}$, $\pi(z) = \sum_{k=1}^{\infty} P_{k-1}(z)\delta_k$ is in $\mathcal{H}$. By construction, $\pi(z)$ is in the one-dimensional space $\mathrm{Ker}(J^* - zI)$. It is also known that, when the deficiency indices are (1,1), J is an entire operator and δ_1 is an entire gauge for J [12, section 3.1.1 and theorem 3.1.2].

Let us find $\xi(z)$ for the operator J. Taking into account (3.2), $\langle \delta_1, \xi(z) \rangle = 1$ and $\langle \delta_1, \pi(\overline{z}) \rangle = 1$ for all $z \in \mathbb{C}$. Then, since both $\pi(\overline{z})$ and $\xi(z)$ are in $\mathrm{Ker}(J^* - \overline{z}I)$ and δ_1 is entire, $\pi(\overline{z}) = \xi(z)$ for all $z \in \mathbb{C}$. Thus, for any φ in $\mathcal{H}$, $\varphi = \sum_{k=1}^{\infty} \varphi_k \delta_k$, the function $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}_{\delta_1}$ is given by

$$\widehat{\varphi}(z) := \langle \pi(\overline{z}), \varphi \rangle = \sum_{k=1}^{\infty} P_{k-1}(z)\varphi_k, \qquad z \in \mathbb{C}.$$

Clearly, if $\widehat{\varphi}(z) \in \widehat{\mathcal{H}}_{\delta_1}, \overline{\widehat{\varphi}(\overline{z})} \in \widehat{\mathcal{H}}_{\delta_1}$. Whence, in virtue of proposition 3, our space $\widehat{\mathcal{H}}_{\delta_1}$ is a de Branges space and then, by proposition 5, δ_1 is real with respect to $C = \Phi_{\delta_1}^{-1}\widehat{C}\Phi_{\delta_1}$ ($\widehat{C}$ is the conjugation in $\widehat{\mathcal{H}}_{\delta_1}$ given in the proof of proposition 5).

Formula (3.4) is written in this case as

$$f(z) = \sum_{x_n \in \mathrm{Sp}(J_\sharp)} \frac{\langle \pi(\overline{z})\pi(x_n)\rangle}{\|\pi(x_n)\|^2} f(x_n)$$
$$= \sum_{x_n \in \mathrm{Sp}(J_\sharp)} \frac{f(x_n)}{\|\pi(x_n)\|^2} \sum_{k=0}^{\infty} P_k(z)P_k(x_n), \qquad z \in \mathbb{C},$$

where $J_\sharp$ is a certain self-adjoint extension of J.

In a different setting, sampling formulae obtained on the basis of Jacobi operators have been studied before [7, 8]. We remark that in [7] the interpolation formulae differ from those obtained by us. In [8], Jacobi operators are treated without using Krein's theory of entire operators.

(2) The entire operator used here has been taken from [12] and is a particular case of an example given by Krein in [22].

Consider a non-decreasing bounded function $s(t)$ such that

$$s(-\infty) = 0 \qquad \text{and} \qquad s(t-0) = s(t).$$

Fix a function defined for any x in the real interval $(-a, a)$ by

$$F(x) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}s(t).$$

In the linear space $\widetilde{\mathcal{L}}$ of continuous functions in $[0, a)$ vanishing in some left neighbourhood of a, we define a sesquilinear form as follows:

$$(g, f) := \int_0^a \int_0^a F(x-t) f(x)\overline{g(t)}\, \mathrm{d}x\, \mathrm{d}t. \tag{5.2}$$

This form is a quasi-scalar product, i.e., the existence of elements f in $\widetilde{\mathcal{L}}$ such that $f \neq 0$ and nevertheless $(f, f) = 0$ is not excluded.

Denote by $\mathcal{D}$ the set of continuously differentiable functions $f \in \widetilde{\mathcal{L}}$ such that $f(0) = 0$ and define in $\mathcal{D}$ the differential operator $\widetilde{A}$ by the rule $\widetilde{A}f := \mathrm{i}f'$. It is not difficult to show that $(g, \widetilde{A}f) = (\widetilde{A}g, f)$ and $\mathcal{D}$ is quasi-dense in $\widetilde{\mathcal{L}}$. Now, proceeding as in [12, section 2.8.2], one defines the space $\mathcal{L}$ as follows:

$$\mathcal{L} = \widetilde{\mathcal{L}} \backslash \widetilde{0}, \qquad \widetilde{0} = \{f \in \widetilde{\mathcal{L}} : (f, f) = 0\}.$$

In $\mathcal{L}$ we define an inner product by

$$\langle \eta, \varphi \rangle := (g, f), \tag{5.3}$$

where φ and η are equivalence classes containing f and g, respectively. Let $\mathcal{H}$ be the completion of $\mathcal{L}$ and consider in it the operator A such that, for the equivalence class φ containing $f \in \mathcal{D}$, $\mathcal{A}\varphi$ is the equivalence class containing $\widetilde{A}f$. It can be shown that A is an entire operator and that

$$\widehat{\varphi}(z) = \langle \xi(z), \varphi \rangle = \int_0^a \mathrm{e}^{\mathrm{i}zt} f(t)\, \mathrm{d}t,$$

where $f \in \varphi$. This identity, together with (5.2) and (5.3), determines $\xi(z)$ completely [12, section 3.2.2].

Note that, in this example, the entire gauge associated with $\xi(z)$ remains unknown. This is not however an issue since the sampling kernel can be computed anyway by resorting to expression (3.5).

6. Concluding remarks

In this work, we have introduced a sampling theory formulated in terms of the theory of representation of regular symmetric operators due to Krein. We have been motivated by some ideas discussed in the context of quantum gravity, and also by realizing that this representation theory seems to be hidden in much of the development in sampling theory found in the literature.

Our main concluding remark is concerned with the 'efficiency' of the sampling formula (3.4). Different notions of efficiency may be defined according to different ways of looking at a sampling formula [23]. Concretely, given a set $\hat{\mathcal{H}}$ of functions:

- One may ask whether a sampling formula exists such that it yields a unique and stable reconstruction of every function in $\widehat{\mathcal{H}}$ from the knowledge of its values at some 'smaller' set of sampling points. This sampling formula would be more efficient for reconstructing functions, since it would require less data to accomplish it, while the stability would ensure control on error.
- Conversely, one may be interested on determining the existence of sampling formulae that allow the encoding of a 'bigger' set of data as samples of some (unique) function in $\widehat{\mathcal{H}}$. In this case, this formula would be more efficient for the purpose of transmitting data.

These questions have been discussed in detail within the context of the WSK interpolation theory (that is, sampling in Paley–Wiener spaces); see for instance [23]. There the notion of efficiency is formulated in terms of the so-called Nyquist rate, which is given by the inverse of the distance between any two consecutive sampling points (which are always equidistant). In [23] it is shown that, for functions of a given Paley–Wiener space, no (stable) reconstruction is possible by sampling at lower than the (associated) Nyquist rate, nor data transmission may be performed by sampling at higher than the Nyquist rate. Thus, in both senses, one may consider the WSK sampling formula as efficient.

Although one may use the Nyquist rate for characterizing the efficiency of (3.4) for particular cases, the more general situation considered in the present work seems to lack a notion similar to that of a Nyquist rate. This is brought about by the fact that the self-adjoint extensions of a generic operator in the class $\mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ do not generally have a uniform distribution of their spectrum. Notably, the self-adjoint extensions of an operator in the subclass of entire operators have a uniform asymptotic density which is upper bounded by a quantity that depends on the indicator diagram associated with the given entire operator; for detail, see [12, section 5.3]. It is worth mentioning that a sort of generalized definition of the Nyquist rate has been already proposed in [14].

Notwithstanding the previous comment, one may attempt the following (rather obvious) approach. Let us consider some operator $A \in \mathrm{Sym}_{\mathrm{R}}^{(1,1)}(\mathcal{H})$ and some appropriate gauge $\mu \in \mathcal{H}$. These choices fix a particular space $\widehat{\mathcal{H}}_\mu$ of functions to be interpolated. Let $\{a_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a set whose elements we would like to use sampling points. Because every real number belongs to the spectrum of a unique self-adjoint extension of A, we have the decomposition

$$\{a_n\}_{n\in\mathbb{Z}} = \bigcup_i S_i, \qquad S_i \subset \mathrm{Sp}(A_i), \qquad A_i^* = A_i \supset A,$$

where furthermore this countable decomposition is unique. Associated with every set S_i, there is a certain closed subspace $\mathcal{H}_i \subset \mathcal{H}$. Then one and only one of the following situations may arise.

(1) The direct sum of $\{\mathcal{H}_i\}$ is a proper subspace of $\mathcal{H}$.
(2) The direct sum of $\{\mathcal{H}_i\}$ yields $\mathcal{H}$, although at least some of the subspaces are not orthogonal to each other.
(3) All the subspaces are pairwise orthogonal and $\bigoplus_i \mathcal{H}_i = \mathcal{H}$.

It is expected that optimal stable sampling is possible only for the last case, which moreover should occur only when $\{a_n\}_{n\in\mathbb{Z}}$ is the spectrum of some self-adjoint extension of A.

Another notion relevant to sampling theory is oversampling, which entails several nice features like the possibility of recovering lost sampling data and the improvement of convergence of interpolation formulae [24]. In the WSK theory, oversampling stems from the fact that Paley–Wiener spaces have an ordering property: the Paley–Wiener space of bandwidth σ is a subspace of that of bandwidth σ' whenever $\sigma < \sigma'$. Since de Branges spaces (of which the Paley–Wiener spaces are particular realizations) also admit a partial ordering [5, chapter 2], it is expected that oversampling gives rise to similar features for the sampling theory associated with entire operators.

The previous remarks indicate several directions for further investigation. We also consider of interest to generalize our sampling theory to other spaces of functions. There are (at least) two ways of doing that by using an approach like the one introduced in this work: on one hand, one may consider Krein's theory of generalized gauges; on the other hand, one may attempt to construct a sampling theory out of a family of finite-rank perturbations of a self-adjoint operator. The latter seems to be specially interesting because it yields a sampling theory for spaces of functions having prescribed singularities on the real line (which might be relevant to applications on quantum gravity for spacetime geometries having singularities). Some results in this direction will be published elsewhere.

Acknowledgments

We express our gratitude to M Ballesteros for drawing our attention to the results of [14] and to A Minkin for useful remarks that led to an improved version of this work. We also thank the referees for valuable comments and a hint to the literature.

References

[1] Akhiezer N I 1965 *The Classical Moment Problem and Some Related Questions in Analysis* (New York: Hafner)
[2] Akhiezer N I and Glazman I M 1993 *Theory of Linear Operators in Hilbert Space* (New York: Dover)
[3] Annaby M H 2003 On sampling theory associated with the resolvents of singular Sturm–Liouville problems *Proc. Amer. Math. Soc.* **131** 1803–12
[4] Boumenir A and Zayed A I 2002 Sampling with a string *J. Fourier Anal. Appl.* **8** 211–31
[5] de Branges L 1968 *Hilbert Spaces of Entire Functions* (Englewood Cliffs, NJ: Prentice-Hall)
[6] de Branges L 1959 Some Hilbert spaces of entire functions *Proc. Amer. Math. Soc.* **10** 840–6
[7] García A G and Hernández-Medina M A 2001 The discrete Kramer sampling theorem and indeterminate moment problems *J. Comput. Appl. Math.* **134** 13–22
[8] García A G and Hernández-Medina M A 2003 Discrete Sturm–Liouville problems, Jacobi matrices and Lagrange interpolation series *J. Math. Anal. Appl.* **280** 221–31
[9] García A G and Hernández-Medina M A 2005 Sampling theory associated with a symmetric operator with compact resolvent and de Branges spaces *Mediterr. J. Math.* **2** 345–56
[10] García A G and Littlejohn L L 2004 On analytic sampling theory *J. Comput. Appl. Math.* **171** 235–46
[11] García A G 2000 Orthogonal sampling formulas: a unified approach *SIAM Rev.* **42** 499–512

[12] Gorbachuk M L and Gorbachuk V I 1997 *M G Krein's Lectures on Entire Operators (Operator Theory: Advances and Applications)* vol 97 (Basel: Birkhaüser)

[13] Kaltenbäck M and Woracek H 1999 Pontryagin spaces of entire functions I *Integr. Equ. Oper. Theory* **33** 34–97

[14] Kempf A 2004 Fields with finite information density *Phys. Rev.* D **69** 124014

[15] Kempf A 2004 Covariant information-density cutoff in curved space-time *Phys. Rev. Lett.* **92** 221301

[16] Kempf A 1999 On the three short-distance structures which can be described by linear operators *Rep. Math. Phys.* **43** 171–7

[17] Kotel'nikov V A 1933 On the carrying capacity of the 'ether' and wire in telecomunications *All-Union Conference on Questions of Communications (Izd. Red. Upr. Svyazi RKKA, Moscow)* (in Russian)

[18] Kramer H P 1959 A generalized sampling theorem *J. Math. Phys.* **38** 68–72

[19] Krein M G 1944 On Hermitian operators with defect numbers one *Dokl. Akad. Nauk SSSR* **43** 339–42 (in Russian)

[20] Krein M G 1944 On Hermitian operators with defect numbers one: II *Dokl. Akad. Nauk SSSR* **44** 143–6 (in Russian)

[21] Krein M G 1944 On one remarkable class of Hermitian operators *Dokl. Akad. Nauk SSSR* **44** 191–5 (in Russian)

[22] Krein M G 1949 Fundamental aspects of the representation theory of Hermitian operators with deficiency indices (m,m) *Ukr. Mat. Zh.* **2** 3–66 (in Russian)

[23] Landau H J 1967 Sampling, data transmission, and the Nyquist rate *Proc. IEEE* **55** 1701–6

[24] Marks R J II 1991 *Introduction to Shannon Sampling and Interpolation Theory* (New York: Springer)

[25] Shannon C E 1949 Communication in the presence of noise *Proc. IRE* **37** 10–21

[26] Simon B 1998 The classical moment problem as a self-adjoint finite difference operator *Adv. Math.* **137** 82–203

[27] Whittaker J M 1929 The 'Fourier' theory of the cardinal function *Proc. Edinburgh. Math. Soc.* **1** 169–76

[28] Zayed A I 1993 A new role of Green's function in interpolation and sampling theory *J. Math. Anal. Appl.* **175** 222–38

[29] Zayed A I, Hinsen G and Butzer P L 1990 On Lagrange interpolation and Kramer-type sampling theorems associated with Sturm–Liouville problems *SIAM J. Appl. Math.* **50** 893–909